Combinations

Inquire: \( C(n,r) = \frac{n!}{r!(n-r)!} \)

Overview

The combination formula is an accurate way to define groups in which the order doesn’t matter. For this lesson, we will use the combination formula to solve examples choosing sub-groups from populations. Doing this without the formula would be time-consuming and probably inaccurate. There will be multiple examples, including choosing side dishes from a restaurant and arranging trophies on a shelf. There will be step-by-step instructions for solutions. This lesson will build your knowledge base in probability and solving equations.

Big Question: If you needed to know how many groups of three you could choose from a group of twelve without using the combination formula, where would you begin?

Watch: Combination Lock?

If you choose a PIN for your bank card, the order of the numbers will certainly matter. However, if you select three of your old T-shirts to use for rags, order will definitely not matter. In the area of probability, the term combination refers to groups in which the order does not matter.

Two types of examples frequently used in calculating combinations include choosing two side dishes from a menu of five at a restaurant or selecting five people out of a group of nine to stand in the front row for a picture. In each of these events, the order within the group does not matter.

The outcome of the formula will be the number of groups that contain the correct number of elements. This tells us how many possible combinations can be made from the data given.

An excellent way to remember the difference between combinations and permutations is to think of a combination lock. The combination must be entered in a specific order, so it should actually be called a permutation lock. If you can remember the permutation lock, you can remember that order does not matter in combinations.

The formula for calculating combinations is: \( C(n,r) = \frac{n!}{r!(n-r)!} \). If we use the values from the restaurant example, we have \( 5!/2!(5-2)! \). Extend the numerator to \( 5 \times 4 \times 3! \) (We leave the 3 in factorial form to cancel with the denominator.) The denominator will be \( 2!(3)! \). After the 3’s are canceled, we have \( 5 \times 4/2! \) and \( 2! = 2 \times 1 \). The solution is 20/2 or 10. There are ten possible groups of two sides.
The number selected is generally less than the total number of the population; however, if all the objects are chosen, the formula will still work. Note that if you choose all the objects, the number of combinations will be one.

Read: Combinations

Overview

Permutations and combinations are usually taught together. They are similar topics with one significant difference. Permutations are lists in which order matters. Combinations are groups in which order does not matter. If you are cooking soup for dinner tonight, it doesn’t matter the exact order in which you add the ingredients. However, if you make a sandwich, the order of ingredients does matter. Soup is a combination of ingredients. A sandwich is a permutation of ingredients.

When to Use Combinations for Counting Events

Use combinations when order does not matter. Consider this: we have four paintings and want to hang three of them on the wall. If the order of the paintings matters, we will use the permutation formula and find 24 permutations.

What if we do not care about the order? We could expect a smaller number because selecting paintings \{1, 2, 3\} would be the same as selecting paintings \{2, 3, 1\}. To find the number of ways to select three of the four paintings, disregarding the order of the paintings, divide the number of permutations by the number of ways to order three paintings.

There are 3! ways to order three paintings: \(3! = 3 \times 2 \times 1 = 6\). So, \(24 \div 6 = 4\) (24 permutations divided by 6 ways to order three paintings). There are four ways to select three of the four paintings. This number makes sense because every time we select three paintings, we do not select one painting. There are four paintings we could choose not to select, so there are four ways to select three of the four paintings.

The basic rule to remember is that permutations are used when the order matters; combinations are used when the order does not matter.

Calculating Combinations

A selection of \(r\) objects from a set of \(n\) objects where the order does not matter can be written as \(C(n,r)\). Just as with permutations, \(C(n,r)\) can also be written as \(^nC_r\). In this case, the general formula is as follows.

Given \(n\) distinct objects, the number of ways to select \(r\) objects from the set is:

\[
C(n, r) = \frac{n!}{r!(n-r)!}
\]

Given a number of options, determine the possible number of combinations by these steps.

1. Identify \(n\) from the given information.
2. Identify \(r\) from the given information.
3. Replace \(n\) and \(r\) in the formula with the given values.
4. Evaluate.
Example 1: Side Dishes

A fast food restaurant offers five side dish options. Your meal comes with two side dishes.

How many ways can you select your side dishes? (We want to choose 2 side dishes from 5 options.)

\[ C(n, r) = \frac{n!}{r!(n-r)!} \]

\[ C(5, 2) = \frac{5!}{2!(5-2)!} = \frac{5!}{2!(3)!} = \frac{5 \times 4 \times 3!}{2 \times 1 \times 3!} = \frac{20}{2} = 10 \]

Notice that we only extended the 5! to 5 x 4 x 3! so we could cancel the 3! in the numerator and the denominator.

Example 2: Picking a Team

We are choosing a team of 3 people from a group of 10. Let \( n = 10 \) and \( r = 3 \). Order doesn’t matter. We just need the number of possible groups of three people. We will use the combination formula.

\[ C(10,3) = \frac{10!}{7!3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \]

This is written in a different style, and the values for \( n \) and \( r \) have already been plugged in. Can you explain why there is a 7! and 3! in the denominator? Answer: Because \( n \) is 10, and \( r \) is 3, \((10-3)! = 7!\)

What became of the 7! in the numerator and denominator? Answer: They canceled out and are not shown.

Skipping steps in solving any equation is not the best practice. It is important in these examples for you to recognize what information is given and what steps are not shown.

Example 3: Video Games

Over the weekend, you are going on vacation, and you are bringing your video game console as well as five of your games. How many ways can you choose the five games to bring if you have 12 games in all?

Let \( n = 12 \) and \( r = 5 \).

\[ C(12,5) = \frac{12!}{5!(12-5)!} = \frac{12!}{5!(7)!} \]

\[ = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{5 \times 4 \times 3 \times 2 \times 1 \times 7!} \]

The 7!s cancel out, leaving:

\[ 12 \times 11 \times 10 \times 9 \times 8 / 5 \times 4 \times 3 \times 2 \times 1 \]

\[ 95,040 / 120 = 792 \]

This example shows the steps of the solution. There are 792 separate groups of five that could be made from the 12 video games. If we were looking for permutations, the number would be much larger.

Reflect Poll: Combinations and Permutations

How do you believe you will most benefit from learning about combinations and permutations?
- I will be able to participate in discussions with other people who use these skills.
- I will be able to go further with my education.
- I will gain confidence because I have been able to learn these skills.
- I will better be able to find employment because of these skills.

Expand: More Formula Examples

Overview

Math joke: Combination locks should really be called “permutation locks,” because the order matters! The next time you see a combination lock, think of it as a permutation lock. If it were truly a “combination,” then it would not matter in what order the numbers were entered. The combinations in this lesson are all examples of situations in which the order of the elements do not matter.

Example 4: The Bus

Twelve people want to ride the bus to the game on Saturday. There is one block of six seats available. How many groups of six could be chosen from the twelve who want to go?

This is a combination example, looking for groups in which order doesn’t matter.

Step 1. Write the formula for combinations.

\[ C(n,r) = \frac{n!}{r!(n-r)!} \]

Step 2. Substitute numbers for letters.

\[ C(12,6) = \frac{12!}{6!(12-6)!} \]

Step 3. Evaluate inside parentheses.

\[ = \frac{12!}{6!(6)!} \]

Step 4. Extend 12!. Cancel when possible.

\[ 12 \times 11 \times 10 \times 9 \times 8 \times 6!/(6!(6)!/6)! \]

This will leave \[ 12 \times 11 \times 10 \times 9 \times 8 \times 7/6 \times 5 \times 4 \times 3 \times 2 \times 1 \]

665,280/720 = 924. There are 924 possible groups or combinations.

Example 5: Car Wash

There are seven cars waiting at the car wash, and there are only three wash bays. How many groups of three can wash their cars at the same time?

\[ C(n,r) = \frac{n!}{r!(n-r)!} \]

\[ C(7,3) = \frac{7!}{3!(7-3)!} \]

\[ = \frac{7!}{3!(4)!} \]

\[ = 7 \times 6 \times 5 \times 4!/3!(4)! \]

\[ = 7 \times 6 \times 5/3 \times 2 \times 1 \]

\[ = 210/6 \]

\[ = 35 \]
There are 35 possible combinations of three cars that can enter the wash bays at the same time. The order in which they enter does not matter.

Lesson Toolbox

Additional Resources and Readings

A one minute clip explaining the basic differences between permutations and combinations simply
  ● Link to resource: https://youtu.be/zRgB3hOIfEMU

A video focusing on understanding when to add and when to multiply with permutations and combinations
  ● Link to resource: https://youtu.be/0NAASclUm4k

A tutorial on combinations also offering examples of when to use the permutation formula and when to use the combination formula
  ● Link to resource: https://youtu.be/PSS3mCS_Ef8

Lesson Glossary

combinations: a group of objects in which order does not matter
permutations: a list of objects in which order does matter

Check Your Knowledge

1. What is the term for the arrangement that selects r objects from a set of n objects when the order of the r objects is not important?
   a. combination
   b. permutation
   c. independent event
   d. dependent event

2. Prior to Reconstruction, enslaved men and women were not permitted the same expressions of If you are choosing three people from a group of ten, what value will you substitute for n?
   a. 13
   b. 7
   c. 3
   d. 10

3. What does 0! equal?
   a. 1
   b. 0
   c. -1
   d. -0

Answer Key:
1. A  2. D  3. A
Citations

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Title: Precalculus. Chapter 11.5 Counting Principles. Openstax. License: CC BY 4.0.
Link to resource: https://cnx.org/contents/_VPq4foj@8.1:MYP1-SMN@12/Counting-Principles