Relations and Functions

Inquire: What’s Your Function?

Overview
Defining, viewing, and applying relationships is a key part of algebra. An example of relationships can be seen in the linear equation \( y = 2x \). Any value of \( y \) will be a doubled value of \( x \). If \( x \) is 5, then \( y \) is 10. If \( x \) is -72, then \( y \) is -144.

If you determine a relation is a function, it can be used to highlight trends or make predictions. A book manufacturing company can relate the cost of printing a book to how many units are made. This would allow the company to maximize its profits.

By the end of the lesson, the student will be able to find the domain and range of a relation, determine if a relation is a function using equations or a vertical line test, and graph various types of functions.

Big Question: How can I determine if a relation is a function?

Watch: A Careful Selection
A relation describes the relationship between two sets of elements and connects inputs to outputs. We actually have relations all around us everyday. Imagine that your stomach rumbles as you look over choices in a vending machine. You haven’t eaten anything all day and could use a snack and drink to hold you over until your next meal. After much thought, you finally spot your selection. You press A3 and receive your snack.

In this case, the button press is your input and the snack is your output. Like a relation, the machine produces the output because it is responding to an input. Every time you press A3, you will get the snack you want. If you change your mind, you could press different buttons to get a different snack.

A special type of relation is called a function. Given that the domain is an input and the range is an output, the formal definition of a function is a relation in which each element of the domain is paired with exactly one element of the range. The words in this definition are exact because functions have two special properties: every input is used and each will only have one result. A function allows a relation to be predictable. It might be limited by its inputs, but you know every output will be matched to a certain input.

In a working vending machine, every button is matched with a snack, and a single button will always provide the same snack. There would be a limit to the inputs and outputs you could have on the machine.
since you can't press a button and get a snack that isn't in the machine. Be careful because some inputs, outputs, or relations are not allowed.

The vending machine would be broken if you pressed a button and it did not give you the snack you wanted. Similarly, a relation is not a function if an input gives two or more different outputs.

Understanding functions will unlock more uses for equations. What input to output relations can you think of?

Read: Relationship Advice

Overview

A relation relates an input to an output. It allows us to observe the connection between two collections of things. Similarly in a vending machine, certain pressed buttons will get you a snack. A button press acts as an input, while the received snack serves as an output.

We can use equations and relations to discuss a relation with special rules: a function. In this lesson, we will define domain and range in a relation, determine if a relation is a function, and find the value of a function.

Domain and Range of Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names. For example, our social security numbers, student ID numbers, email addresses, phone numbers, and birthdays are matched to our names. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and the corresponding student ID numbers are likely in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student’s ID number, we call this a relation.

\[(\text{Student name, Student ID #})\]

The set of student names, or all the x-values, are called the domain of the relation. The set of student ID numbers paired with these students, or all the y-values, are the range (sometimes called codomain) of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is called a relation. The following relations will be written in set notation. This is where elements are written in a list inside curly brackets and separated with commas.

Example 1: For the relation \{((1,1), (2,4), (3,9), (4,16), (5,25))\}, find domain and range.
Domain: All the x-values are on the left side of the coordinates. The domain is \{1, 2, 3, 4, 5\}.
Range: All the y-values are on the right side of the coordinates. The range is \{1, 4, 9, 16, 25\}.

A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.
Example 2: Use the mapping shown below to A) list the ordered pairs as a relation B) find the domain and C) find the range.

A) The arrows match the person to their birthday, so we can write ordered pairs using names as the x-values and birthdays as the y-values. This produces the following relation:

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}. The order will not matter as long as you have all of the pairs.

B) The domain is the set of all x-values of the relation. In this example, the domain will be all of the listed names.
{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

C) The range is the set of all y-values of the relation. In this example, the range will be all of the birthdays.
{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

Relations and Functions

A special type of relation, called a function, occurs extensively in mathematics. A function is a relation that assigns each element in its domain exactly one element in the range. For each ordered pair in the relation, each x-value is matched with only one y-value.

The birthday example from Example 2 can help us understand this definition. Every person has a birthday, but no one has two birthdays. However, it is okay for two people to share a birthday, as Danny and Stephen share July 24th and June and Liz share August 2nd. Since each person has exactly one birthday, the relation in Example 1 is a function.

Example 3: Determine if the relations below are functions.

A) {(-3,27), (-2,8), (-1,1), (0,0), (1,1), (2,8), (3,27)}
Each x-value is matched with only one y-value. So, this relation is a function.
Both Lydia and Marty have two phone numbers. So each x-value is not matched with only one y-value. This relation is not a function.

**Value of a Function**

It is very convenient to name a function; most often we name functions f, g, h, F, G, or H. In any function, each x-value from the domain has a corresponding y-value in the range. For the function f, we write this range value y as f(x). This is called function notation and is read 'f of x' or 'the value of f at x'. In this case, the parentheses do not indicate multiplication. Function notation is a useful way to write functions without writing out every input and output.

We call x the independent variable as it can be any value in the domain. We call y the dependent variable as its value depends on x. For instance, if we let x = 2 in the equation y = 4x - 5, we can find out that y = 3. The equation can be rewritten as the function f(x) = 4x - 5. When we let x = 2 in this function, it looks like this:

\[
f(x) = 4x - 5 \\
f(2) = 4(2) - 5 \\
f(2) = 3
\]

Notice that we replaced f(x) with f(2) to show that we are plugging 2 into the x-value. Also, the final line can be read "if x = 2, then y = 3". This process of finding the value of f(x) for a given value of x is called evaluating the function.

Let's model an everyday situation using functions.

Example 4: Sylvia has 75 unread emails in her account. This number grows by 10 unread emails a day. The function \( N(t) = 75 + 10t \) represents the relation between the number of emails, N, and the time, t, measured in days.

A) Determine the independent and dependent variable.
The number of unread emails is a function of the number of days. The number of unread emails, N, depends on the number of days, t. Therefore, the variable N, is the dependent variable and the variable t is the independent variable.

B) Find N(5). Explain what this result means.
N(5) means t = 5. This evaluates to the following:
N(t) = 75 + 10t
N(5) = 75 + 10(5)
N(5) = 75 + 50
N(5) = 125

Since 5 is the number of days, N(5), is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

Reflect Poll: Coordinates, Maps, or Equations?

We have shared three ways to display a relation. Which would you most likely use to display the relationship between how many hours you work and how much you earn if your job pays $8 a hour?

- A set of coordinates. Example: {(1,8), (2,16), (3,24)...}
- Mapping. Example: 1 → 8, 2 → 16, 3 → 24...
- An equation. Example: $ = 8h; where $ is money earned and h is hours worked

Expand: Graphs are FUNctional

Overview

Graphs are a useful way to represent relations and functions. We can plot coordinates or use methods from linear equations to graph them. This is a great way to visualize the relationship between domain and range or determine if a relation is a function.

Vertical Line Test

A relation is a function if every element of the domain has exactly one value in the range, like the relation defined by the equation y = 2x − 3. This example is shown in the table of solutions below.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>(x, y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-7</td>
<td>(-2, -7)</td>
</tr>
<tr>
<td>-1</td>
<td>-5</td>
<td>(-1, -5)</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
<td>(0, -3)</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(3, 3)</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>(4, 5)</td>
</tr>
</tbody>
</table>

If we look at the graph, each vertical dashed line only intersects the line at one point. This makes sense as, in a function, for every x-value there is only one y-value.
If the vertical line hit the graph twice, the x-value would be mapped to two y-values, and so the graph would not represent a function. This idea is the namesake for the **vertical line test**. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph at only one or no points. If any vertical line intersects the graph at more than one point, the graph does not represent a function.

Example 1: Determine if the following graphs are functions.

Graphs A and C are functions because it is impossible to draw a vertical line through two points of the graph.
Graph B and D are not functions since there are multiple cases where you could draw a vertical line through two points.

Graphing Basic Functions

Similar to linear equations ($y = mx + b$), a linear function is a function that can be written as $f(x) = mx + b$, where $m$ is the slope and $b$ is the $y$-intercept. All lines, except for vertical lines, are functions.

Example 2: Graph $f(x) = -2x - 4$.

Since this is a linear function, the slope ($m$) is -2 (or $-2/1$) and the $y$-intercept ($b$) is -4. After plotting a point on the $y$-axis where $y = -4$, we use the slope of $-2/1$ (which means “2 down and 1 right”). This is shown in the graph below.

Another common function is called the constant function and its equation is written as $f(x) = b$, where $b$ is any real number. If we replace the $f(x)$ with $y$, we get $y = b$. We recognize this as the horizontal line whose $y$-intercept is $b$. The graph of the function $f(x) = b$, is also the horizontal line whose $y$-intercept is $b$.

Example 3: Graph $f(x) = 4$.

This is a horizontal line through $y = 4$. This graph is shown below.
Graphing Unfamiliar Functions

The next three functions we will look at are not linear functions, meaning when graphed, they will not be a line. The only method we have to graph these functions is point plotting. Because these are unfamiliar functions, we must make sure to choose several positive and negative values, as well as 0, for our x-values.

Example 4: Graph \( f(x) = x^2 \).

We choose x-values \{-3, -2, -1, 0, 1, 2, 3\}, substitute them in, create a chart, and graph them as shown. As a reminder, the work for two substitutions will be done here. The results of the rest of the substitutions will be in the table of solutions below.

\[
\begin{align*}
  f(x) &= x^2 \\
  f(1) &= (1)^2 \\
  f(1) &= 1 \\
  f(x) &= x^2 \\
  f(-2) &= (-2)^2 \\
  f(-2) &= 4 \\
\end{align*}
\]

This is a square function. The graph is the shape of a parabola.

Example 5: Graph \( f(x) = x^3 \).

As with example 4, we will create a table of solutions using the same positive and negative values, as well as 0. The work for two substitutions will be done here, and the results of the rest of the substitutions will be in the table of solutions below.

\[
\begin{array}{c|c|c}
  x & f(x) = x^3 & (x, f(x)) \\
  \hline
  -3 & 9 & (-3, 9) \\
  -2 & 4 & (-2, 4) \\
  -1 & 1 & (-1, 1) \\
  0 & 0 & (0, 0) \\
  1 & 1 & (1, 1) \\
  2 & 4 & (2, 4) \\
  3 & 9 & (3, 9) \\
\end{array}
\]
f(x) = x^3
f(2) = (2)^3
f(2) = 8

f(x) = x^3
f(0) = (0)^3
f(0) = 0

This is a cube function.

Example 6: f(x) = √x

This function takes the square root of values. For this function, we need to carefully select values as we can’t square negative numbers and some squares give us irrational numbers. Since we will be taking the square root, we choose numbers that are perfect squares, to make our work easier. We substitute them in and then create a chart.
Lesson Toolbox

Additional Resources and Readings
A game that practices evaluating functions, identifying domain and range, and determining if a relation is a function

A short video describing different ways to determine if a relation is a function
  ● Link to resource: https://www.youtube.com/watch?v=YwuRIZ1Alu4

A video focusing on what a function is and how it is used
  ● Link to resource: https://www.youtube.com/watch?v=52tpYl2tTqk

Lesson Glossary

domain: all the x-values in the ordered pairs of a relation
evaluating the function: the process of finding the value of f(x) for a given value of x
function: a relation that assigns to each element in its domain exactly one element in the range; Domain is the independent variable and range is the dependant variable
function notation: the way a function is written; most popularly written as f(x) which is read ‘f of x’
mapping: sometimes used to show a relation; the arrows show the pairing of the elements of the domain with the elements of the range
range: all the y-values in the ordered pairs of a relation
relation: the set of ordered pairs (x,y) that records a matching; All the x-values are the domain, and all the y-values are the range
set notation: elements are written in a list inside curly brackets and separated with commas
vertical line test: test to assess a function, based on whether a vertical line intersects the graph at more than one point

Check Your Knowledge

1. Write the domain as a list of numbers separated by commas and no spaces (Example: 10,20,30).

2. Write the range as a list of numbers separated by commas and no spaces (Example: 10,20,30).

3. Is the set a function?
   a. Yes
   b. No

Answer Key:
1. 1,2,3,4,5  2. 4,8,12,16,20  3. A